



Lee–Wick theories at high temperature

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ABSTRACT

An extension of the standard model, the Lee–Wick standard model, based on ideas of Lee and Wick was recently introduced. It does not contain quadratic divergences in the Higgs mass and hence solves the hierarchy puzzle. The Lee–Wick standard model contains new heavy Lee–Wick resonances at the TeV scale that decay to ordinary particles. In this Letter we examine the behavior of Lee–Wick resonances at high temperature. We argue that they contribute negatively to the energy density ρ and pressure p and at temperatures much greater than their mass M their $\mathcal{O}(T^4)$ contributions to ρ and p cancel against those of the ordinary (light) particles. The remaining $\mathcal{O}(M^2 T^2)$ contributions are positive and result in an equation of state that approaches $w = 1$ from below as $T \rightarrow \infty$.

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1. Introduction

Recently ideas proposed by Lee and Wick [1,2] were used to extend the standard model so that it does not contain quadratic divergences in the Higgs mass [3]. Higher derivative kinetic terms are added for each of the standard model fields. They improve the convergence of Feynman diagrams and result in a theory where there are no quadratically divergent radiative corrections to the Higgs mass. The higher derivative terms give rise to propagators with new poles that are massive resonances. These Lee–Wick (LW) resonances have wrong-sign kinetic terms which naively give rise to unacceptable instabilities and violations of unitarity. Lee and Wick [1,2] and Cutkosky et al. (CLOP) [4] proposed a way of defining the integrations that arise in Feynman diagrams so that the theory is unitary, Lorentz invariant, and free of instabilities. However, there is acausal behavior caused by the unusual location of poles in the propagators. Physically this acausality is associated with the future boundary condition needed to forbid the exponentially growing modes. As long as the masses and widths of the LW resonances are large enough, this acausality does not manifest itself on macroscopic scales and is not in conflict with scattering ex-

periments. Various aspects of this model [5,6], its extensions [7,8] and of Lee–Wick theories in general [9–12] have been explored in the recent literature. Collider phenomenology [13–15], constraints from electroweak precision measurement [8,16,17], chiral symmetry breaking mechanism [18] and the cosmology of theories with higher derivatives [19] have also been studied. Since we will be studying the very high temperature limit, it is important that amplitudes describing scattering of longitudinally polarized massive vector bosons present in non-Abelian Lee–Wick gauge theory do not grow with energy and, hence, satisfy the constraints imposed by perturbative unitarity [20].

In this Letter we examine the high temperature behavior of Lee–Wick theories, including the LW standard model. In these theories the S-matrix can be calculated in perturbation theory using the prescriptions of Lee and Wick and CLOP. It is unclear whether a functional integral formulation of LW theory exists, so a computation of finite temperature effects solely based on the known S-matrix is desired. The formalism of Dashen, Ma and Bernstein (DMB) [21] expresses the thermodynamic grand potential in terms of the S-matrix and we apply it to LW theories to deduce the pressure and energy density for these theories at finite temperature. Although previous analyses have argued that in scattering experiments no acausal effects persist to macroscopic scales, it is interesting to examine whether this is possible when multiple scattering effects play a role. This is the case for thermal equilibrium and we explore the propagation of sound waves in a gas

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consisting of ordinary and Lee–Wick particles. We find that in such a gas, at a large (but finite) temperature, sound waves propagate at a speed less than light.

In the next section we review scattering in a simple scalar Lee–Wick theory. Section 3 uses the DMB formalism to calculate, in this toy model, the energy density and pressure at thermal equilibrium. At high temperatures $T \gg M$ we find that the LW resonance contributes minus what an ordinary particle of mass M would. We use our results to conclude that as $T \rightarrow \infty$ the speed of sound approaches $c_s = 1$ from below. This gives in the limit $T \rightarrow \infty$ a speed of sound equal to the speed of light and is the largest value consistent with causal propagation of classical sound waves in the gas. Concluding remarks are made in Section 4.

2. A toy model

In this section we introduce a simple Lee–Wick theory with a single self-interacting real scalar field. In addition to the standard kinetic term there is a higher derivative term. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{2M^2} (\partial^2 \hat{\phi})^2 - \frac{1}{2} m^2 \hat{\phi}^2 - \frac{1}{3!} g \hat{\phi}^3, \quad (1)$$

so the propagator of $\hat{\phi}$ in momentum space is given by

$$D_F(p) = \frac{i}{p^2 - p^4/M^2 - m^2}. \quad (2)$$

For $M \gg m$, this propagator has poles at $p^2 = m^2$ and also at $p^2 = M^2$. Thus, the propagator describes more than one degree of freedom.

We can make these new degrees of freedom manifest in the Lagrangian density in a simple way. First, let us introduce an auxiliary scalar field $\tilde{\phi}$, so that we can write the theory as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{2} m^2 \hat{\phi}^2 - \tilde{\phi} \partial^2 \hat{\phi} + \frac{1}{2} M^2 \tilde{\phi}^2 - \frac{1}{3!} g \hat{\phi}^3. \quad (3)$$

Next, we define $\phi = \hat{\phi} + \tilde{\phi}$. In terms of this variable, after integrating by parts, the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{1}{2} M^2 \tilde{\phi}^2 - \frac{1}{2} m^2 (\phi - \tilde{\phi})^2 - \frac{1}{3!} g (\phi - \tilde{\phi})^3. \quad (4)$$

In this form, it is clear that there are two kinds of scalar fields: a normal scalar field ϕ and a new field $\tilde{\phi}$, which we will refer to as a LW field. The sign of the quadratic Lagrangian of the LW field is opposite to the usual sign so one may worry about stability of the theory, even at the classical level. We will return to this point. If we ignore, for simplicity, the mass m , the propagator of $\tilde{\phi}$ is given by

$$\tilde{D}_F(p) = \frac{-i}{p^2 - M^2}. \quad (5)$$

The LW field is associated with a non-positive definite norm on the Hilbert space, as indicated by the unusual sign of its propagator. Consequently, if this state were to be stable, unitarity of the S -matrix would be violated. However, as emphasized by Lee and Wick, unitarity and Lorentz invariance can be preserved provided that $\tilde{\phi}$ may decay. This is natural in the theory described by Eq. (4) because $\tilde{\phi}$ is heavy and can decay into two ϕ particles.

In the presence of the mass m , there is a mixing between the scalar field ϕ and the LW scalar $\tilde{\phi}$. We can diagonalize this mixing without spoiling the diagonal form of the derivative terms by performing a hyperbolic rotation of the fields: $\phi = \phi' \cosh \theta + \tilde{\phi}' \sinh \theta$,

$\tilde{\phi} = \phi' \sinh \theta + \tilde{\phi}' \cosh \theta$. This transformation diagonalizes the Lagrangian if

$$\tanh 2\theta = \frac{-2m^2/M^2}{1 - 2m^2/M^2}. \quad (6)$$

A solution for the angle θ exists provided $M > 2m$. The Lagrangian density (4) describing the system becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2} m'^2 \phi'^2 - \frac{1}{2} \partial_\mu \tilde{\phi}' \partial^\mu \tilde{\phi}' + \frac{1}{2} M'^2 \tilde{\phi}'^2 - \frac{1}{3!} g' (\phi' - \tilde{\phi}')^3, \quad (7)$$

where m' and M' are the masses of the diagonalized fields and $g' = (\cosh \theta - \sinh \theta)^3 g$. In what follows we assume that $M \gg m$, so that $g' \simeq g$.

Introducing the LW fields makes the physics of the theory clear. There are two fields; the heavy LW scalar decays to lighter scalars. At loop level, the presence of the heavier scalar improves the convergence of loop graphs at high energy consistent with our expectations from the higher derivative form of the theory.

Loop corrections to the two point function of the LW field play a crucial role. Near $p^2 = M^2$ and at small g the $\tilde{\phi}$ – ϕ mixing can be neglected and the full LW $\tilde{\phi}$ propagator and its perturbative expansion are given by

$$\begin{aligned} \tilde{D}_F(p) &= \frac{-i}{p^2 - M^2} + \frac{-i}{p^2 - M^2} [i\Sigma(p^2)] \frac{-i}{p^2 - M^2} + \dots \\ &= \frac{-i}{p^2 - M^2 - \Sigma(p^2)}. \end{aligned} \quad (8)$$

Note that, unlike for ordinary scalars, there is a minus sign in front of the self-energy $\Sigma(p^2)$ in the denominator. The pole mass shift of the LW scalar coming from the radiative corrections is $+\Sigma(M^2)$. This sign is significant; for example, from a one-loop computation we see that the imaginary part of the self energy is

$$\text{Im } \Sigma(p^2) = \frac{g^2}{32\pi} \theta(p^2 - 4m^2) \sqrt{1 - \frac{4m^2}{p^2}}. \quad (9)$$

Therefore the propagator develops a pole for $\text{Im}(p^2) > 0$. In the narrow width approximation the propagator for the LW field is

$$D_{\text{LW}} = \frac{-i}{p^2 - M^2 + iM\Gamma}, \quad (10)$$

where

$$\Gamma = -\frac{g^2}{32\pi M} \sqrt{1 - \frac{4m^2}{M^2}}. \quad (11)$$

This width differs in sign from widths of the usual unstable particles we encounter. Strictly speaking the propagator has an additional pole at $p^2 = M^2 + iM\Gamma$ and a cut over the real axis; however, the effect of these two terms in the below calculation of the pressure and energy density at finite temperature cancel one another so we ignore them.

Using the Lee and Wick and CLOP prescriptions the S -matrix in this theory is unitary and Lorentz invariant on the space of physical ordinary ϕ particles. The contour of integration over p^0 in this prescription does not lead to any instability, as one may naively guess from the negative sign in the width, but instead leads to apparently acausal behavior. This has been extensively discussed in the literature [1,10,22].

The LW resonance $\tilde{\phi}$ is unstable and therefore does not appear in the initial or final states of the S -matrix. This is similar to the case of the W -boson of the standard model, which does not appear in initial or final states of S -matrix elements because it is

unstable.¹ Nonetheless, the $\tilde{\phi}$ resonance impacts the scattering of ordinary ϕ particles $\phi(p_1) + \phi(p_2) \rightarrow \phi(p'_1) + \phi(p'_2)$, particularly near the kinematic point $(p_1 + p_2)^2 = M^2$.

Writing $S = 1 - iT$, the T matrix can be computed using the standard Feynman techniques, modified appropriately for Lee–Wick theories. DMB introduce a closely related quantity $\mathcal{T}(E)$ that has two particle matrix elements

$$\langle \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 | \mathcal{T}(E) | \mathbf{p}_1, \mathbf{p}_2 \rangle = (2\pi) \delta(E - E_1 - E_2) (2\pi)^3 \delta^3(\mathbf{P} - \tilde{\mathbf{P}}) \mathcal{M}(E), \quad (12)$$

where $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, $\tilde{\mathbf{P}} = \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2$ and \mathcal{M} is essentially the usual invariant matrix element. For center of mass energies near M the invariant matrix element is given (in the narrow resonance approximation) by²

$$\mathcal{M}(E) = -\frac{1}{2} \frac{g^2}{E^2 - \mathbf{P}^2 - M^2 + iM\Gamma}. \quad (13)$$

Note that this differs from scattering via the exchange of an ordinary (i.e., not LW) resonance by an overall minus sign and the fact that Γ given by Eq. (11) is negative.

3. The pressure and energy density in thermal equilibrium

The grand partition function Ω at zero chemical potential is defined by

$$\Omega = \frac{1}{\beta} \ln \text{Tr} e^{-\beta H}, \quad (14)$$

where $\beta = 1/kT$ and the trace is over all physical states in the theory. In our toy model the physical states are the ϕ particle states but not states that contain a LW resonance. From Ω one can calculate the thermal equilibrium pressure p and energy density ρ in the usual fashion using formulas

$$p = -\frac{\Omega}{V}, \quad \rho = -\frac{\partial(\beta p)}{\partial \beta}, \quad (15)$$

where V is the volume of the system.

DMB derive the following expression for the grand potential

$$\Omega = \Omega_0 - \frac{1}{\beta} \int dE e^{-\beta E} \frac{1}{4\pi i} \left(\text{Tr} A S(E)^{-1} \overleftrightarrow{\frac{\partial}{\partial E}} S(E) \right)_c, \quad (16)$$

where c denotes that only connected diagrams are taken into account. In Eq. (16) the S -matrix is given by $S(E) = 1 - iT(E)$, Ω_0 is the free particle grand potential and A is an operator that sums over permutations of the identical particles in the trace with the appropriate minus signs for fermions. Using the relation between S and T this becomes

$$\Omega = \Omega_0 - \frac{1}{\beta} \int dE e^{-\beta E} \frac{1}{4\pi i} \times \left[\text{Tr} A \left(-i \frac{\partial}{\partial E} [\mathcal{T}(E) + \mathcal{T}(E)^\dagger] + \mathcal{T}(E)^\dagger \overleftrightarrow{\frac{\partial}{\partial E}} \mathcal{T}(E) \right) \right]_c. \quad (17)$$

To evaluate Ω in the toy model introduced in the previous section, we begin by evaluating the part of Ω that comes from the contribution of two particle ϕ states. It is convenient to use the phase space relation

$$\begin{aligned} & \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \\ &= \int d^3 P \int \frac{d^3 p'_1}{(2\pi)^3 2E'_1} \int \frac{d^3 p'_2}{(2\pi)^3 2E'_2} \delta^3(\mathbf{p}'_1 + \mathbf{p}'_2) \frac{\omega}{E}, \end{aligned} \quad (18)$$

where the primed variables are the center of mass momenta and energies, $\omega = E'_1 + E'_2$ and $E = E_1 + E_2 = \sqrt{\omega^2 + \mathbf{P}^2}$. In order to calculate the first term of the integral in Eq. (17) we notice that³

$$\text{Tr} \frac{\partial}{\partial E} \mathcal{T}(E) = \frac{\partial}{\partial E} \text{Tr} \mathcal{T}(E), \quad (19)$$

so the expression we need to evaluate is

$$\begin{aligned} \text{Tr} \mathcal{T}(E) &= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \\ &\times (2\pi)^3 \delta^3(0) (2\pi) \delta(E - E_1 - E_2) \mathcal{M}(E). \end{aligned} \quad (20)$$

Using $(2\pi)^3 \delta^3(0) = V$ and the phase space relation (18) gives

$$\begin{aligned} \text{Tr} \mathcal{T}(E) &= V \int \frac{d^3 P}{(2\pi)^3} \mathcal{M}(E) \int \frac{d^3 p'_1}{(2\pi)^3 2E'_1} \int \frac{d^3 p'_2}{(2\pi)^3 2E'_2} \\ &\times \delta^3(\mathbf{p}'_1 + \mathbf{p}'_2) \frac{\omega}{E} (2\pi)^4 \delta(E - E_1 - E_2). \end{aligned} \quad (21)$$

Now, recall that $P^\mu = (E, \mathbf{P})$ is the total energy–momentum four-vector of the states in the trace while $(\omega, \mathbf{0})$ is the corresponding energy–momentum four-vector in the center of mass frame. They are related by a boost with a Lorentz gamma factor $\gamma = E/\omega$. One can go from one set of variables to the other. Using the relation $E^2 = \omega^2 + \mathbf{P}^2$ we have that

$$\frac{\omega}{E} \delta(E - E_1 - E_2) = \delta(\omega - E'_1 - E'_2) \quad (22)$$

and so the integrations over $d^3 p'_1 d^3 p'_2$ become the standard two body phase space integration. We arrive at the result

$$\text{Tr} \frac{\partial}{\partial E} \mathcal{T}(E) = V \frac{\partial}{\partial E} \int \frac{d^3 P}{(2\pi)^3} \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{\omega^2}} \mathcal{M}(E). \quad (23)$$

Finally, we change from the variable E to ω in all other places. Using

$$dE \frac{\partial}{\partial E} = d\omega \frac{\partial}{\partial \omega} \quad (24)$$

and interchanging the order of the two integrations we get

$$\begin{aligned} & \int dE e^{-\beta E} \text{Tr} \frac{\partial}{\partial E} \mathcal{T}(E) \\ &= \int \frac{d^3 P}{(2\pi)^3} \int d\omega e^{-\beta \sqrt{\omega^2 + \mathbf{P}^2}} \frac{\partial}{\partial \omega} \left(\frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{\omega^2}} \mathcal{M}(\omega) \right), \end{aligned} \quad (25)$$

where

$$\mathcal{M}(\omega) = -\frac{1}{2} \frac{g^2}{\omega^2 - M^2 + iM\Gamma}. \quad (26)$$

The term with $\mathcal{T}(E)^\dagger$ gives the same contribution to Ω but with $\mathcal{M}(\omega)$ substituted by $\mathcal{M}^*(\omega)$. In calculating the second term of the integral in Eq. (17) we use the same methods as previously. We need to evaluate

$$\begin{aligned} & \text{Tr} \mathcal{T}(E)^\dagger \overleftrightarrow{\frac{\partial}{\partial E}} \mathcal{T}(E) \\ &= V \int \prod_{i=1}^2 \frac{d^3 p_i}{(2\pi)^3 2E_i} \int \prod_{i=1}^2 \frac{d^3 \hat{p}_i}{(2\pi)^3 2\hat{E}_i} (2\pi)^3 \delta^3(\mathbf{P} - \hat{\mathbf{P}}) \\ &\times \left[(2\pi) \delta(E - E_1 - E_2) \mathcal{M}^*(E) \right] \\ &\times \overleftrightarrow{\frac{\partial}{\partial E}} \left[(2\pi) \delta(E - \hat{E}_1 - \hat{E}_2) \mathcal{M}(E) \right], \end{aligned} \quad (27)$$

¹ One difference is that there are no poles in amplitudes associated with the W -boson two point function, just a cut that is represented as a pole in the narrow width approximation. However, for the Lee–Wick $\tilde{\phi}$ resonance there is actually a pair of negative residue poles in the complex plane in addition to the usual cut.

² There is an extra factor of $1/2$ associated with identical particles that we have chosen to put in \mathcal{M} rather than in phase space integrations.

³ For simplicity we drop the subscript c .

where $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, $\hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2$ and the factor $V(2\pi)^3 \delta^3(\mathbf{P} - \hat{\mathbf{P}})$ came from the momentum delta functions in the definition (12) of $\mathcal{T}(E)$. In the c.m. frame we have

$$\begin{aligned} \text{Tr } \mathcal{T}(E)^\dagger \frac{\overleftrightarrow{\partial}}{\partial E} \mathcal{T}(E) &= V \int \frac{d^3 P}{(2\pi)^3} \int \frac{d^3 \hat{P}}{(2\pi)^3} (2\pi)^3 \delta^3(\mathbf{P} - \hat{\mathbf{P}}) \\ &\times \left[\int \prod_{i=1}^2 \frac{d^3 p'_i}{(2\pi)^3 2E'_i} \delta^3(\mathbf{p}'_1 + \mathbf{p}'_2) \right. \\ &\times \frac{\omega}{E} (2\pi)^4 \delta(E - E_1 - E_2) \mathcal{M}^*(E) \left. \right] \\ &\times \frac{\overleftrightarrow{\partial}}{\partial E} \left[\int \prod_{i=1}^2 \frac{d^3 \hat{p}'_i}{(2\pi)^3 2\hat{E}'_i} \delta^3(\hat{\mathbf{p}}'_1 + \hat{\mathbf{p}}'_2) \right. \\ &\times \frac{\hat{\omega}}{E} (2\pi)^4 \delta(E - \hat{E}_1 - \hat{E}_2) \mathcal{M}(E) \left. \right] \\ &= V \int \frac{d^3 P}{(2\pi)^3} \left[\frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{\omega^2}} \mathcal{M}^*(E) \right] \\ &\times \frac{\overleftrightarrow{\partial}}{\partial E} \left[\frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{\omega^2}} \mathcal{M}(E) \right]. \end{aligned} \quad (28)$$

Putting all this together and neglecting the mass m of the ordinary scalars, we arrive at the LW contribution to the grand potential in the form

$$\begin{aligned} \Omega_{\text{LW}} &= -\frac{V}{\beta} \int \frac{d^3 P}{(2\pi)^3} \int d\omega e^{-\beta\sqrt{\omega^2 + \mathbf{P}^2}} \frac{1}{4\pi i} \\ &\times \left[-i \frac{\partial}{\partial \omega} \left(\frac{\mathcal{M}(\omega)}{8\pi} + \frac{\mathcal{M}^*(\omega)}{8\pi} \right) \right. \\ &\left. + \frac{\mathcal{M}^*(\omega)}{8\pi} \frac{\overleftrightarrow{\partial}}{\partial \omega} \frac{\mathcal{M}(\omega)}{8\pi} \right] + \dots, \end{aligned} \quad (29)$$

where the ellipses denote the terms from summing over permutations which basically are multiple insertions of the two body state. Performing the differentiations and using the explicit formulas for \mathcal{M} and Γ this becomes

$$\begin{aligned} \Omega_{\text{LW}} &= -\frac{V}{\beta} \int \frac{d^3 P}{(2\pi)^3} \int d\omega e^{-\beta\sqrt{\omega^2 + \mathbf{P}^2}} \\ &\times \left[\frac{2}{\pi} \frac{\omega M \Gamma}{(\omega^2 - M^2)^2 + M^2 \Gamma^2} \right] + \dots. \end{aligned} \quad (30)$$

Recall that for the LW resonance Γ is negative. The above formula is the same as one would get for scattering through an ordinary resonance except in that case Γ is positive. Therefore, in the narrow LW resonance approximation

$$\frac{2}{\pi} \frac{\omega M \Gamma}{(\omega^2 - M^2)^2 + M^2 \Gamma^2} \rightarrow -\delta(\omega - M). \quad (31)$$

Hence, the contribution to the grand potential from the single LW resonance is

$$\Omega_{\text{single LW}} = +\frac{V}{\beta} \int \frac{d^3 P}{(2\pi)^3} e^{-\beta\sqrt{M^2 + \mathbf{P}^2}}. \quad (32)$$

This is precisely what one would expect from a stable particle of mass M , except for the overall plus sign instead of a minus. Note that the narrow width approximation is valid provided the prefactor in the integral over ω is slowly varying. This will be the case provided $\beta|\Gamma| \ll 1$.

Thus far we have only included two particle states in the calculation of the trace for the grand potential, with a resonant S -matrix from Eq. (13). Contributions to the trace from states with more than two particles introduce multiple resonance amplitudes. These have two effects [23]: they modify the width, as expected for decays in a thermal bath, and they convert the exponential factor to the usual Bose–Einstein logarithm. In the narrow width approximation the modified width is still narrow, that is, proportional to g^2 , and hence it still vanishes as $g \rightarrow 0$. The Bose–Einstein logarithm arises from considering multiple resonance graphs that are connected only because of the permutation operator A . This gives

$$\Omega_{\text{LW}} = -\frac{V}{\beta} \int \frac{d^3 P}{(2\pi)^3} \ln(1 - e^{-\beta\sqrt{M^2 + \mathbf{P}^2}}), \quad (33)$$

which is minus the contribution of a boson of mass M to the ideal gas grand potential. Note that this result is valid for arbitrarily large temperatures, which is not the case for non-elementary (composite) resonances [24].

Since in the narrow width approximation the LW resonance contributes minus what an ordinary scalar particle of mass M would, in our toy model the LW contribution to the energy density is

$$\rho_{\text{LW}} = -\left[\frac{\pi^2 (kT)^4}{30} - \frac{M^2 (kT)^2}{24} \right] + \dots, \quad (34)$$

while the contribution to the pressure is

$$p_{\text{LW}} = -\left[\frac{\pi^2 (kT)^4}{90} - \frac{M^2 (kT)^2}{24} + \frac{M^3 (kT)}{12\pi} \right] + \dots. \quad (35)$$

Here the ellipses stand for terms of order $\ln(T)$ at most and are less important than those explicitly displayed when $T \gg M$. Adding these to the positive contributions from the ordinary scalar (whose mass m we neglect)

$$\rho_{\text{ordinary}} = \frac{\pi^2 (kT)^4}{30}, \quad p_{\text{ordinary}} = \frac{\pi^2 (kT)^4}{90}, \quad (36)$$

gives

$$\rho = \rho_{\text{ordinary}} + \rho_{\text{LW}} = \frac{M^2 (kT)^2}{24} + \dots \quad (37)$$

and

$$p = p_{\text{ordinary}} + p_{\text{LW}} = \frac{M^2 (kT)^2}{24} - \frac{M^3 (kT)}{12\pi} + \dots. \quad (38)$$

A similar analysis holds for theories with a left-handed fermion and its LW partner. This is most easily seen in the higher derivative formulation of the theory. In the auxiliary field formulation one usually introduces left- and right-handed LW fermions but one of these is dependent on the other through the equations of motion. In that case⁴

$$\rho_{\text{LW}}^{\text{F}} = -\left[\frac{7\pi^2 (kT)^4}{120} - \frac{M^2 (kT)^2}{24} \right] + \dots \quad (39)$$

and

$$p_{\text{LW}}^{\text{F}} = -\left[\frac{7\pi^2 (kT)^4}{360} - \frac{M^2 (kT)^2}{24} \right] + \dots, \quad (40)$$

where, similarly to the boson case, the ellipses denote terms of order $\ln(T)$ at most. The absence of a term linear in T for the fermion pressure and energy density is as in the normal case.

⁴ We take into account a factor of $2s + 1$ in the formula for the grand partition function, where s is the spin.

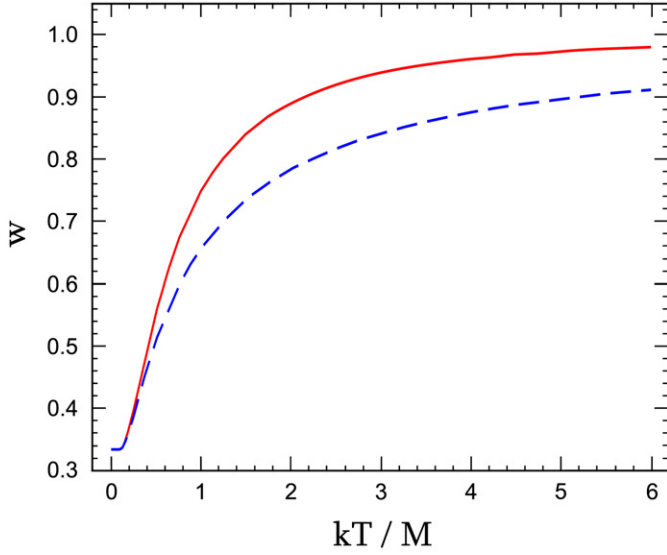


Fig. 1. Factor w in the equation of state $p = w\rho$ as a function of kT/M for fermions (solid) and bosons (dashed).

Adding the LW fermion contribution to the ordinary fermion energy density and pressure

$$\rho_{\text{ordinary}}^F = \frac{7\pi^2(kT)^4}{120}, \quad p_{\text{ordinary}}^F = \frac{7\pi^2(kT)^4}{360}, \quad (41)$$

gives

$$\rho^F = \rho_{\text{ordinary}}^F + \rho_{\text{LW}}^F = \frac{M^2(kT)^2}{24} + \dots \quad (42)$$

and

$$p^F = p_{\text{ordinary}}^F + p_{\text{LW}}^F = \frac{M^2(kT)^2}{24} + \dots \quad (43)$$

From the above formulas for the pressure and energy density one can calculate the factor w in the equation of state $p = w\rho$ to be

$$w = 1 - \frac{2}{\pi} \frac{M}{kT} + \mathcal{O}\left[\frac{\ln(T)}{T^2}\right] \quad \text{for bosons,}$$

$$w = 1 + \mathcal{O}\left[\frac{\ln(T)}{T^2}\right] \quad \text{for fermions.} \quad (44)$$

In Fig. 1 we plot $w = p/\rho$ for both cases as a function of kT/M . The value $w = 1/3$ for small temperatures should not be puzzling since for $T \rightarrow 0$ the LW contribution is suppressed by the Boltzmann weight factors and only the ordinary particles contribute to the grand potential. More interesting is the value $w = 1$ at high temperatures, which may have implications to the early universe cosmology. Cosmology with equation of state $w = 1$ has been investigated in the context of holographic cosmology for a medium consisting of a black hole emulsion [25–29].

The speed of sound c_s can be calculated using the formula

$$c_s = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{dp}{dT} / \frac{d\rho}{dT}}. \quad (45)$$

Taking into account higher order correction terms we arrive at

$$c_s = 1 - \frac{1}{2\pi} \frac{M}{kT} + \mathcal{O}\left[\frac{\ln(T)}{T^2}\right] \quad \text{for bosons,}$$

$$c_s = 1 + \mathcal{O}\left[\frac{\ln(T)}{T^2}\right] \quad \text{for fermions.} \quad (46)$$

Fig. 2 shows the plot of c_s for fermions and bosons as a function of kT/M . The speed of sound increases from $1/\sqrt{3}$ at $T = 0$ to 1

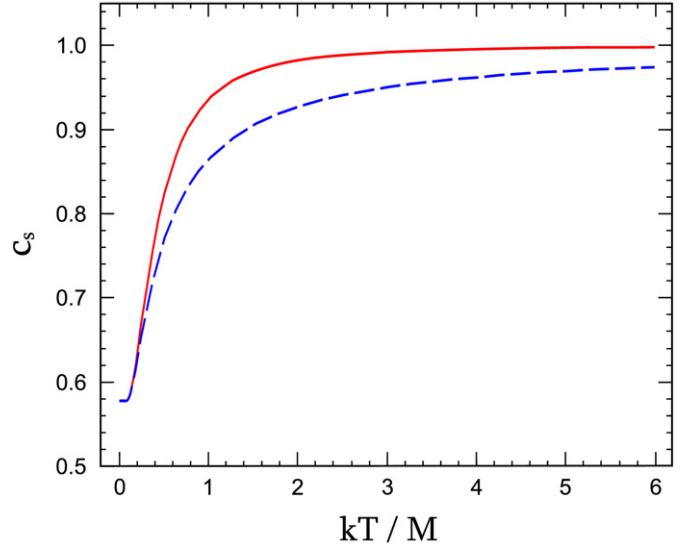


Fig. 2. Speed of sound $c_s = \sqrt{dp/d\rho}$ as a function of kT/M for fermions (solid) and bosons (dashed).

as $T \rightarrow \infty$, which is equal to the speed of light. Hence, the propagation of classical sound waves in the relativistic gas is causal.

In the LW standard model there is a LW partner for every ordinary particle. Hence, at high temperatures above all the LW masses the pressure and energy density are approximately proportional to $M^2 T^2$, which implies the same equation of state $w = 1$. This gives a universe with density $\rho \sim a(t)^{-6}$ and the scale factor $a(t) \propto t^{1/3}$.

4. Conclusions

We have studied in this Letter Lee–Wick theories at high temperature. Making use of the S -matrix formulation of statistical mechanics presented in DMB [21], we calculated the grand thermodynamical potential for a gas of Lee–Wick resonances in the boson and fermion case. We found that the contribution of Lee–Wick resonances to the energy density and pressure is negative for high temperatures.

Next, we considered a gas of both ordinary and Lee–Wick particles. We found that for high temperatures (much greater than the mass of the resonance itself) the contributions of the Lee–Wick resonances to the pressure and energy density cancel against those of the normal particles at leading order in temperature (i.e., $\mathcal{O}(T^4)$). We confirmed this for both fermions and bosons. We found that the remaining $\mathcal{O}(M^2 T^2)$ contribution is positive and identical for the pressure and energy density. This led us to the equation of state $w = 1$ for $T \rightarrow \infty$. In applications to big bang cosmology this yields a scale factor of the universe $a(t) \propto t^{1/3}$.

The quantity $c_s = \sqrt{dp/d\rho}$ corresponds to the speed of sound in the medium. We checked that c_s is less than 1 in the whole temperature range, thus causality is not violated. It seems interesting to investigate the cosmological implications of the equation of state $w = 1$, especially for the propagation of fluctuations in the early universe.

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References

- [1] T.D. Lee, G.C. Wick, Nucl. Phys. B 9 (1969) 209.
- [2] T.D. Lee, G.C. Wick, Phys. Rev. D 2 (1970) 1033.
- [3] B. Grinstein, D. O'Connell, M.B. Wise, Phys. Rev. D 77 (2008) 025012, arXiv:0704.1845 [hep-ph].
- [4] R.E. Cutkosky, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, Nucl. Phys. B 12 (1969) 281.
- [5] J.R. Espinosa, B. Grinstein, D. O'Connell, M.B. Wise, Phys. Rev. D 77 (2008) 085002, arXiv:0705.1188 [hep-ph].
- [6] T.R. Dulaney, M.B. Wise, Phys. Lett. B 658 (2008) 230, arXiv:0708.0567 [hep-ph].
- [7] I. Antoniadis, E. Dudas, D.M. Ghilencea, JHEP 0803 (2008) 045, arXiv:0708.0383 [hep-th].
- [8] C.D. Carone, R.F. Lebed, Phys. Lett. B 668 (2008) 221, arXiv:0806.4555 [hep-ph].
- [9] B. Grinstein, D. O'Connell, Phys. Rev. D 78 (2008) 105005, arXiv:0801.4034 [hep-ph].
- [10] B. Grinstein, D. O'Connell, M.B. Wise, arXiv:0805.2156 [hep-th].
- [11] A. van Tonder, arXiv:0810.1928 [hep-th].
- [12] C.D. Carone, R.F. Lebed, JHEP 0901 (2009) 043, arXiv:0811.4150 [hep-ph].
- [13] T.G. Rizzo, JHEP 0706 (2007) 070, arXiv:0704.3458 [hep-ph].
- [14] F. Krauss, T.E.J. Underwood, R. Zwicky, Phys. Rev. D 77 (2008) 015012, arXiv:0709.4054 [hep-ph].
- [15] T.G. Rizzo, JHEP 0801 (2008) 042, arXiv:0712.1791 [hep-ph].
- [16] E. Alvarez, L. Da Rold, C. Schat, A. Szykman, JHEP 0804 (2008) 026, arXiv:0802.1061 [hep-ph].
- [17] T.E.J. Underwood, R. Zwicky, arXiv:0805.3296 [hep-ph].
- [18] E. Gabrielli, Phys. Rev. D 77 (2008) 055020, arXiv:0712.2208 [hep-ph].
- [19] Y.F. Cai, T.t. Qiu, R. Brandenberger, X.m. Zhang, arXiv:0810.4677 [hep-th].
- [20] B. Grinstein, D. O'Connell, M.B. Wise, Phys. Rev. D 77 (2008) 065010, arXiv:0710.5528 [hep-ph].
- [21] R. Dashen, S.K. Ma, H.J. Bernstein, Phys. Rev. 187 (1969) 345; R. Dashen, S.K. Ma, H.J. Bernstein, Phys. Rev. A 6 (1972) 851, Erratum.
- [22] S. Coleman, in: Erice 1969, Ettore Majorana School On Subnuclear Phenomena, Academic Press, New York, 1970, pp. 282–327.
- [23] R.F. Dashen, R. Rajaraman, Phys. Rev. D 10 (1974) 694.
- [24] R.F. Dashen, R. Rajaraman, Phys. Rev. D 10 (1974) 708.
- [25] T. Banks, W. Fischler, hep-th/0111142.
- [26] T. Banks, W. Fischler, Phys. Scr. T 117 (2005) 56, hep-th/0310288.
- [27] T. Banks, W. Fischler, hep-th/0405200.
- [28] T. Banks, W. Fischler, L. Mannelli, Phys. Rev. D 71 (2005) 123514, hep-th/0408076.
- [29] T. Banks, W. Fischler, hep-th/0412097.